$0 \Rightarrow f\left(\frac{x}{16}\right) = f(x)$; and so, by induction, $f\left(\frac{x}{2^n}\right) = (-1)^n f(x)\ \forall\ n \in \mathbb{N}$. Suppose f is continuous at 0. Since $\frac{x}{2^n} \to 0$, $f\left(\frac{x}{2^n}\right) \to f(0) = 0$. Hence $(-1)^n f(x) \to 0$, and so $f(x) = 0$. We have just shown that if $f(x)+f(2x) = 0\ \forall\ x \in \mathbb{R}$ and f is continuous at 0, then f is identically 0 on $\mathbb{R}$. For a noncontinuous example, first note that $f(2^n x) = (-1)^n f(x) \forall\ n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}$. Let $f(1) = 1$, let $f(2^n) = f\left(\frac{1}{2^n}\right) = (-1)^n f(1) = (-1)^n\ \forall\, n \in \mathbb{N}$, and let $f(x) = 0$ for all other x in $\mathbb{R}$. Then $f(x) + f(2x) = 0\ \forall\ x \in \mathbb{R}$, and f is not continuous at 0.

One last comment. In our opinion Exercise 4.5.12 is hard; we therefore suggest discretion (evaluate the sophistication of your students) before assigning this exercise.

4.1 Continuous Functions

1. Let $c \in \mathbb{R}$, let $\varepsilon > 0$, and let $\delta = \varepsilon$. For $x \in \mathbb{R}$ with $|x - c| < \delta$, by Corollary 2.1, $|f(x) - f(c)| = ||x| - |c|| \leq |x - c| < \delta = \varepsilon$. Therefore, f is continuous at c. Since c is an arbitrary element of $\mathbb{R}$, f is continuous on $\mathbb{R}$.

2. Let $c \neq 0$. <u>To show</u>: f is continuous at c. First note that for $x \neq 0$, $|f(x) - f(c)| = \left|\frac{1}{x} - \frac{1}{c}\right| = \frac{|x - c|}{|c|\,|x|}$. If $|x| > \frac{|c|}{2}$, then $|f(x) - f(c)| < \frac{2\,|x-c|}{|c|^2} = \frac{2\,|x-c|}{c^2}$. Let $\varepsilon > 0$ and let $\delta = \min\left\{\frac{|c|}{2}, \frac{c^2\varepsilon}{2}\right\}$. Let $x \in \mathbb{R}$ with $|x - c| < \delta$. Then $|c| - |x| \leq |x - c| < \delta \leq \frac{|c|}{2}$, and so $|x| > |c| - \frac{|c|}{2} = \frac{|c|}{2}$. Therefore, $|f(x) - f(c)| < \frac{2\,|x-c|}{c^2} < \frac{2\delta}{c^2} \leq \frac{2}{c^2}\cdot\frac{c^2\varepsilon}{2} = \varepsilon$, and so f is continuous at c.
[Alternatively, one could first show that f is continuous on $(0, \infty)$ by an argument similar to that given above. Noting that $g(x) = -x$ is continuous on $(-\infty, 0)$, then $f \circ g$ is continuous on $(-\infty, 0)$ by Proposition 4.3. Since $f = -(f \circ g)$ on $(-\infty, 0)$, f is continuous on $(-\infty, 0)$ by Proposition 4.2.]

3. f is continuous at $c \neq 0$ by Propositions 4.2 and 4.3 and Exercise 2. For instance, $\sin\frac{1}{x}$ is the composition of two continuous functions when $x \neq 0$. <u>To show</u>: f is continuous at 0. Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Let

$x \in \mathbb{R}$ with $|x| = |x - 0| < \delta$. For $x \neq 0$, $|f(x) - f(0)| = \left|x \sin \frac{1}{x}\right| = |x| \left|\sin \frac{1}{x}\right| \leq |x| < \delta = \varepsilon$; of course, $|f(0) - f(0)| = 0 < \varepsilon$. Therefore, f is continuous at 0.

4. For $f(c) < h$, let $\varepsilon = h - f(c) > 0$. Since f is continuous at c, by Definition 4.1, there is a neighborhood U of c such that $x \in U \cap D \Rightarrow f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon)$. Hence, $x \in U \cap D \Rightarrow f(x) < f(c) + \varepsilon = h$. For $f(c) > h$, let $\varepsilon = f(c) - h > 0$. As above, $\exists$ a neighborhood U of c such that $x \in U \cap D \Rightarrow f(x) > f(c) - \varepsilon = h$.

5. Let $\varepsilon = 1$. Since f is continuous at c, $\exists$ a neighborhood U of c such that $x \in U \cap D \Rightarrow |f(x) - f(c)| < 1$. By Corollary 2.1, $|f(x)| - |f(c)| \leq |f(x) - f(c)|$. Hence, $x \in U \cap D \Rightarrow |f(x)| < 1 + |f(c)|$, and so f is bounded on $U \cap D$ by $1 + |f(c)|$.

6. By Proposition 4.1, g is the composition of two continuous functions; hence g is continuous on D by Proposition 4.3.

7. For $n \in \mathbb{N} \cup \{0\}$ and real numbers $a_0, a_1, \ldots, a_n$, a polynomial f is given by $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$. Proposition 4.2 implies that f is continuous on $\mathbb{R}$. For instance, x^k is the product of the continuous function $g(x) = x$ with itself k times.

8. A rational function f is given by $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials. By Exercise 7, p and q are both continuous on $\mathbb{R}$. Proposition 4.2 then implies that f is continuous at x if $q(x) \neq 0$. Since the polynomial q has only a finite number of real roots, f is discontinuous only at a finite set of real numbers, possibly the empty set.

9. We have $f, g : D \to \mathbb{R}$ with f and g both continuous at $c \in D$. <u>To show</u>: fg is continuous at c. For $x \in D$,

$$\begin{aligned} |f(x)g(x) - f(c)g(c)| &\leq |f(x)g(x) - f(x)g(c)| + |f(x)g(c) - f(c)g(c)| \\ &= |f(x)|\,|g(x) - g(c)| + |g(c)|\,|f(x) - f(c)|. \quad (1) \end{aligned}$$

By Exercise 5, f is bounded on a neighborhood of c intersect D. So $\exists$ $M > 0$ and $\delta_1 > 0$ such that $x \in D$ and $|x - c| < \delta_1 \Rightarrow |f(x)| \leq M$. Let $\varepsilon > 0$. Since g is continuous at c, $\exists$ $\delta_2 > 0$ such that $x \in D$ and $|x - c| < \delta_2 \Rightarrow |g(x) - g(c)| < \frac{\varepsilon}{2M}$; since f is continuous at c, $\exists$ $\delta_3 > 0$

such that $x \in D$ and $|x-c| < \delta_3 \Rightarrow |f(x)-f(c)| < \dfrac{\varepsilon}{2(1+|g(c)|)}$. ($g(c)$ could be 0, so we added the 1.) Let $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$. Then $x \in D$ and $|x-c| < \delta$ implies by (1) that

$$\begin{aligned} |f(x)g(x) - f(c)g(c)| &< M \cdot \frac{\varepsilon}{2M} + |g(c)| \cdot \frac{\varepsilon}{2(1+|g(c)|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, fg is continuous at c.
For $k \in \mathbb{R}$, $h(x) = k$ is continuous on D, and so $hf = kf$ is continuous at c.
Since g is continuous at c, $-g = (-1) \cdot g$ is continuous at c. Hence, $f - g = f + (-g)$ is continuous at c.

10. From the hint,

$$h(x) = \max\{f(x), g(x)\} = \frac{1}{2}\left[f(x) + g(x) + |f(x) - g(x)|\right]$$

and

$$k(x) = \min\{f(x), g(x)\} = \frac{1}{2}\left[f(x) + g(x) - |f(x) - g(x)|\right].$$

Proposition 4.2 and Exercise 1 imply that both h and k are continuous on D.

4.2 Continuity and Sequences

1. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ with $f(A) \subset B$. Let $c \in A$ with f continuous at c and g continuous at $f(c)$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A with $x_n \to c$. By Theorem 4.1, $f(x_n) \to f(c)$. Since $(f(x_n))_{n\in\mathbb{N}}$ is a sequence in B which converges to $f(c)$, Theorem 4.1 implies that $g(f(x_n)) \to g(f(c))$; and, therefore, $g \circ f$ is continuous at c.

2. For $c \in \mathbb{Q}$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R} \setminus \mathbb{Q}$ with $x_n \to c$. Then $f(x_n) \to 0 \neq f(c)$. For $c \in \mathbb{R} \setminus \mathbb{Q}$, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q}$ with $y_n \to c$. Then $f(y_n) \to 1 \neq f(c)$. By Theorem 4.1, f is not continuous at any c in $\mathbb{R}$.

3. Draw a picture. <u>To show</u>: f is continuous at 0. Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Let $x \in \mathbb{R}$ with $|x| = |x - 0| < \delta$. Then $|f(x) - f(0)| = |f(x)| \leq$

$|x| < \delta = \varepsilon$, and so f is continuous at 0.
Let $c \neq 0$. To show: f is discontinuous at c. For $c \in \mathbb{Q}$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R} \setminus \mathbb{Q}$ with $x_n \to c$. Then $f(x_n) \to 0 \neq c = f(c)$. For $c \in \mathbb{R} \setminus \mathbb{Q}$, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q}$ with $y_n \to c$. Then $f(y_n) = y_n \to c \neq 0 = f(c)$. By Theorem 4.1, f is not continuous at any $c \neq 0$.

4. Draw a picture. To show: f is continuous at $\frac{1}{2}$. Let $\varepsilon > 0$ and let $\delta = \varepsilon$. Let $x \in \mathbb{R}$ with $\left|x - \frac{1}{2}\right| < \delta$. Then $\left|f(x) - f\left(\frac{1}{2}\right)\right| = \left|f(x) - \frac{1}{2}\right| = \left|x - \frac{1}{2}\right| < \delta = \varepsilon$, and so f is continuous at $\frac{1}{2}$. (Note that for $x \in \mathbb{R} \setminus \mathbb{Q}$, $f(x) - \frac{1}{2} = 1 - x - \frac{1}{2} = \frac{1}{2} - x$.)
Let $c \neq \frac{1}{2}$. To show: f is discontinuous at c. For $c \in \mathbb{Q}$, let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}\setminus\mathbb{Q}$ with $x_n \to c$. Then $f(x_n) = 1 - x_n \to 1 - c \neq c = f(c)$. For $c \in \mathbb{R} \setminus \mathbb{Q}$, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q}$ with $y_n \to c$. Then $f(y_n) = y_n \to c \neq 1 - c = f(c)$. By Theorem 4.1, f is not continuous at any $c \neq \frac{1}{2}$.

5. This is similar to Example 4.6. Let $f(x) = \cos\left(\frac{1}{x}\right)$ for $x \neq 0$. Let $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{\frac{\pi}{2} + 2n\pi}$ $\forall n \in \mathbb{N}$. Then $x_n \to 0$ and $y_n \to 0$, but $f(x_n) = 1$ and $f(y_n) = 0$ $\forall\, n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} f(x_n) \neq \lim\limits_{n\to\infty} f(y_n)$, f cannot be continuously extended to 0.

6. This is similar to Proposition 4.4. Let $x \in \mathbb{R} \setminus \mathbb{Q}$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Q}$ with $x_n \to x$. Then $f(x) = \lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} c = c$.

7. $f(x) = \frac{1}{x}$ is continuous on $(0,1)$ and $\left(\frac{1}{n}\right)_{n=2}^{\infty}$ is a Cauchy sequence in $(0,1)$. Since $\left(f\left(\frac{1}{n}\right)\right)_{n=2}^{\infty} = (n)_{n=2}^{\infty}$ is unbounded, $\left(f\left(\frac{1}{n}\right)\right)_{n=2}^{\infty}$ is not Cauchy. (Uniform continuity (Section 4.5) is needed to guarantee that the image of a Cauchy sequence is Cauchy.)

8. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $(0,\infty) \cap \mathbb{R} \setminus \mathbb{Q}$ with $x_n \to 0$. Then $f(x_n) \to 0 \neq f(0)$, and so f is not continuous at 0.